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SPLIT SQUARE AND SPLIT CARPET AS EXAMPLES OF NON-METRIZABLE IFS ATTRACTORS

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Abstract. We define two non-metrizable compact spaces and show that they are attractors of iterated function systems. Both of them, the split square and the split carpet, are constructed using the product of split intervals.

1. Introduction. The theory of iterated function systems (IFS) defined on general topological spaces is fairly well developed, cf. [2, 6, 7, 12, 15, 16, 18, 19, 20]. In that regard, various concepts of attractors have been introduced. However, the examples of IFSs that are defined on non-metrizable spaces and admit an attractor seem to be very scarce, e.g., [4, Example 6], [17, Example 3.8]. In principle, to construct IFSs with non-metrizable attractors, one could use point transitive dynamical systems (in particular, minimal dynamical systems); see [5, Example 2] for a specific construction which works in general. However, no efficient characterization of spaces admitting transitive (minimal) systems is known, even in the case of metrizable spaces, and the advance of the subject is quite scattered, see for instance [8]. Moreover, the number of standard examples of non-metrizable

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spaces that are compact and separable (a necessary condition for an attractor of an IFS [4, Proposition 5]) is rather small, e.g., [10].

The aim of this work is to provide two examples of IFSs with non-metrizable attractors. They are inspired by the IFS in [4, Example 6], which acts on the split interval (also called double arrow space [10] or two arrows space [11]).

In section 2 we present a construction and properties of the first example, the split square. We show that it is the attractor of an iterated function system constructed in section 3. The second example, called split carpet, is described in the last section.

2. Split square. A *split square* is the set

$$\begin{aligned}\mathbb{Q} = [0,1) \times (0,1] \times \{1\} \,\cup\, (0,1] \times (0,1] \times \{2\} \\ \cup\, (0,1] \times [0,1) \times \{3\} \,\cup\, [0,1) \times [0,1) \times \{4\}.\end{aligned}$$

We endow it with the topology generated by the basis of sets

$$\begin{aligned}Q(a,b;c,d) := [a,b) \times (c,d] \times \{1\} \,\cup\, (a,b] \times (c,d] \times \{2\} \\ \cup\, (a,b] \times [c,d) \times \{3\} \,\cup\, [a,b) \times [c,d) \times \{4\},\end{aligned} \tag{2.1}$$

$0 \le a < b \le 1$, $0 \le c < d \le 1$; see Fig. 1. It is customary to set $Q(a,b;c,d) := \emptyset$ when $a \ge b$ or $c \ge d$. One can easily see that (2.1) forms a basis of some topology. Indeed,

$$Q(a,b;c,d) \cap Q(a',b';c',d') = Q(a \vee a', b \wedge b'; c \vee c', d \wedge d'),$$

where $r \vee s := \max(r,s)$, $r \wedge s := \min(r,s)$. In particular, the above intersection is empty when $a \vee a' \ge b \wedge b'$ or $c \vee c' \ge d \wedge d'$.

The basic open sets (2.1) are clopen thanks to the equality

$$\mathbb{Q} \setminus Q(a,b;c,d) = Q(0,1;0,c) \cup Q(0,1;d,1) \cup Q(0,a;0,1) \cup Q(b,1;0,1).$$

Therefore, $\mathbb{Q}$ is zero-dimensional; see Theorem 2.2 for more properties of $\mathbb{Q}$.

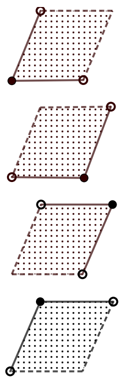

Figure 1. The shape of $\mathbb{Q}$ and basic neighbourhoods.

It should be remarked that a space nearly identical to the split square was considered by T. Banakh in [1] under the same name. Banakh's split square differs from $\mathbb{Q}$ by "edges" as it is a product of two split intervals with uncut "isolated corners"; see Proposition 2.1.

Note also that each slice $\mathbb{Q}\cap\big([0,1]^2\times\{t\}\big)$, $t=1,2,3,4$, bears the topology induced from the Sorgenfrey plane (a product of two Sorgenfrey lines); see Fig. 1.

What is crucial about $\mathbb{Q}$ is that it is the topological product of two split intervals. Let $\mathbb{I} := ((0,1]\times\{0\}) \ \cup \ ([0,1)\times\{1\})$ be the *split interval*. The topology on $\mathbb{I}$ is generated by a basis of sets of two forms

$$\begin{aligned} I_0(a,b) &:= ((a,b]\times\{0\}) \ \cup \ ((a,b)\times\{1\}),\\ I_1(a,b) &:= ((a,b)\times\{0\}) \ \cup \ ([a,b)\times\{1\}), \end{aligned}$$

cf. [11, Exercise 3.10.C, chap. 3, p. 212]. We find it more convenient to work with the basis formed by the scaled split intervals

$$I(a,b) := ((a,b]\times\{0\}) \ \cup \ ([a,b)\times\{1\}),$$

where $0\le a<b\le 1$. To see the equivalence of the two bases, one just observes that $I(a,b)=I_0(a,b)\cup I_1(a,b)$, $I_0(a,b)=\bigcup_{n\in\mathbb{N}} I(a+(b-a)/n,b)$, $I_1(a,b)=\bigcup_{n\in\mathbb{N}} I(a,b-(b-a)/n)$.

Proposition 2.1. *The split square $\mathbb{Q}$ is homeomorphic to $\mathbb{I}^2$, the product of two split intervals.*

Proof. Let us consider the product

$$\begin{aligned} \mathbb{I}^2 = (0,1]\times\{0\}\times(0,1]\times\{0\}&\\ \cup \ (0,1]\times\{0\}\times[0,1)\times\{1\}&\\ \cup \ [0,1)\times\{1\}\times(0,1]\times\{0\}&\\ \cup \ [0,1)\times\{1\}\times[0,1)\times\{1\}&. \end{aligned}$$

Define $h\colon \mathbb{Q}\to\mathbb{I}^2$ by

$$h(x,y,t) := \begin{cases} (x,1,y,0) & \text{when } t=1,\\ (x,0,y,0) & \text{when } t=2,\\ (x,0,y,1) & \text{when } t=3,\\ (x,1,y,1) & \text{when } t=4, \end{cases} \tag{2.2}$$

for $(x,y,t)\in\mathbb{Q}$. The map h is a homeomorphism. Indeed, h is bijective, and

$$h(Q(a,b;c,d)) = I(a,b)\times I(c,d) \tag{2.3}$$

for $a<b, c<d$ in $[0,1]$. ∎

Basic properties of $\mathbb{Q}$ are gathered below.

Theorem 2.2. *The split square $\mathbb{Q}$ is a non-metrizable zero-dimensional compact separable Hausdorff space.*

Proof. Follows from Proposition 2.1 and properties of the split interval, cf. [11]. ∎

We finish this section with a lemma providing a wide class of continuous functions on $\mathbb{I}$ and $\mathbb{Q}$.

The product of two maps $f\colon X\to X$ and $g\colon Y\to Y$ is denoted by $f\times g : X\times Y\to X\times Y$. Recall that $f\times g\,(x,y) := (f(x),g(y))$ for $(x,y)\in X\times Y$. The restriction of $f\times g$ to a subset of $X\times Y$ will be also denoted by $f\times g$. If f and g are continuous and $X\times Y$

bears the product topology, then $f \times g$ is continuous. The identity map will be denoted by id.

LEMMA 2.3. *Let $g_1, g_2\colon [0,1] \to [0,1]$ be increasing, continuous maps in the Euclidean sense.*

(a) The map $g_1 \times \mathrm{id}$ defined on $\mathbb{I}$ is continuous in the topology of split interval.
(b) The map $g_1 \times g_2 \times \mathrm{id}$ defined on $\mathbb{Q}$ is continuous in the topology of split square.

Proof. Note that the preimages via increasing maps g_i preserve shapes of intervals. Therefore,

$$(g_1 \times \mathrm{id})^{-1}(I(a,b)) = I(a',b'),$$
$$(g_1 \times g_2 \times \mathrm{id})^{-1}(Q(a,b;c,d)) = Q(a',b';c',d'),$$

for $a < b$ and $c < d$ in $[0,1]$, where $a' = g_1^{-1}(a)$, $b' = g_1^{-1}(b)$, $c' = g_2^{-1}(c)$, $d' = g_2^{-1}(d)$. ■

REMARK 2.4. The base $I(a,b) \times I(c,d)$, $a < b$, $c < d$, in $\mathbb{I}^2$ corresponds to the base (2.1) in $\mathbb{Q}$ according to the formula (2.3). Similarly, in the context of Lemma 2.3, the product of two maps $g_1 \times \mathrm{id}$ and $g_2 \times \mathrm{id}$ on $\mathbb{I}$ stays in the natural correspondence with the map $g_1 \times g_2 \times \mathrm{id}$ on $\mathbb{Q}$. Namely,

$$h \circ (g_1 \times g_2 \times \mathrm{id}) = (g_1 \times \mathrm{id}) \times (g_2 \times \mathrm{id}),$$

where h is given by (2.2).

3. IFSs on the split square. In this work by an *iterated function system* (briefly IFS) we understand a finite collection of continuous maps $f_i : X \to X$, $i \in I$, acting on a Hausdorff topological space X, e.g., [4, 6]. We write $\mathcal{F} = (X; f_i : i \in I)$. Let $\mathcal{K}(X)$ be a hyperspace of nonempty compact subsets of X. We endow $\mathcal{K}(X)$ with the *Vietoris topology* whose subbasic open sets are of the following two forms:

$$V^+ := \{S \in \mathcal{K}(X) : S \subseteq V\} = \mathcal{K}(V),$$
$$V^- := \{S \in \mathcal{K}(X) : S \cap V \neq \emptyset\},$$

where V runs over all open sets in X, e.g., [13]. Note that a sequence of compact sets is convergent in the Vietoris sense if and only if it is convergent in the *upper Vietoris topology*, which is generated by the sets V^+, and the *lower Vietoris topology*, which is generated by the sets V^-, e.g., [9, Sec. 10]. We associate with an IFS $\mathcal{F}$ the *Hutchinson operator* $F : \mathcal{K}(X) \to \mathcal{K}(X)$. We will write F^n for the n-fold composition of F and, for simplicity, $F^n(x)$ instead of $F^n(\{x\})$.

A set $A \in \mathcal{K}(X)$ is a *strict attractor* of the IFS $\mathcal{F}$ provided there exists an open $U \supseteq A$ such that for all $K \in \mathcal{K}(U)$,

$$F^n(K) \underset{n\to\infty}{\longrightarrow} A$$

in the Vietoris sense. The maximal open set U satisfying the above property is called a *basin* of the attractor A. If one can take $U = X$, then A is said to have a *full basin*. In particular, $F(A) = A$ because F is continuous in the Vietoris topology, cf. [3]. It should be noted that a strict attractor A is necessarily a separable space, e.g., [4, Proposition 5].

Further on, we use letters $\mathcal{F}$, $\mathcal{G}$ for the IFSs and F, G for the corresponding Hutchinson operators.

For example, the unit interval $[0,1]$ is a strict attractor of the IFS $(\mathbb{R}; g_i : i = 0,1)$ and its restriction

$$\mathcal{G} := ([0,1]; g_i : i = 0,1), \tag{3.1}$$

where

$$g_i(x) := \frac{x+i}{2}. \tag{3.2}$$

Similarly, the square $[0,1]^2$ is a strict attractor of the IFS $(\mathbb{R}^2; g_i \times g_j : i,j \in \{0,1\})$, with g_i given by (3.2).

It was shown in [4, Example 6] that the split interval $\mathbb{I}$ is a strict attractor of a suitably defined IFS on $\mathbb{I}$. We modify this construction below by omitting one of the maps in the original IFS.

Example 3.1. Let $\mathcal{F} = (\mathbb{I}; f_i : i = 0,1)$, where $f_i(x,t) := \left(\frac{x+i}{2}, t\right)$ for $(x,t) \in \mathbb{I}$. Then $\mathbb{I}$ is a strict attractor of $\mathcal{F}$ with full basin.

Fix $(x,t) \in \mathbb{I}$. The upper Vietoris convergence $F^n(x,t) \to \mathbb{I}$ is obvious. To show the lower Vietoris convergence, let $V \subseteq \mathbb{I}$ be an arbitrary nonempty open set; that is $V \cap \mathbb{I} \neq \emptyset$. Then $V \supseteq I(a,b)$ for some $0 \leq a < b \leq 1$. Denote $U := (a,b)$. Since $[0,1]$ is the attractor of the IFS $\mathcal{G}$ defined in (3.1), (3.2), there exists $n_0 \in \mathbb{N}$ such that $U \cap G^n(x) \neq \emptyset$ for $n \geq n_0$. Therefore

$$V \cap F^n(x,t) \supseteq U \times \{t\} \cap G^n(x) \times \{t\} \neq \emptyset$$

for $n \geq n_0$, due to the relations: $F^n(x,t) = G^n(x) \times \{t\}$, and $V \supseteq U \times \{t\}$. To finish the reasoning, one needs to ensure the Vietoris convergence $F^n(K) \to \mathbb{I}$ for every $K \in \mathcal{K}(\mathbb{I})$, not only for $K = \{(x,t)\}$. This is immediate by the squeezing argument, since

$$F^n(x,t) \subseteq F^n(K) \subseteq \mathbb{I},$$

where (x,t) is a point picked anyhow from the set K.

Let $\mathcal{F}_1$ be an IFS on X_1 and let $\mathcal{F}_2$ be an IFS on X_2. We define the *product IFS* as follows $\mathcal{F}_1 \times \mathcal{F}_2 := (X_1 \times X_2; f_1 \times f_2 : f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2)$. This can be generalized to several factors. The projections $\pi_k : X_1 \times X_2 \to X_k$, $k = 1,2$, are defined as $\pi_k(x_1,x_2) := x_k$.

It turns out that a finite product of strict attractors is again a strict attractor. We formulate a suitable lemma for two factors.

Lemma 3.2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two IFSs that are defined on X_1 and X_2, respectively, and which have strict attractors A_1 and A_2 with basins U_1 and U_2. Then $A_1 \times A_2$ is a strict attractor of the product system $\mathcal{F}_1 \times \mathcal{F}_2$ with a basin that contains $U_1 \times U_2$.*

Proof. For every $K \in \mathcal{K}(U_1 \times U_2)$, $(x_1,x_2) \in K$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} F_1^n(x_1) \times F_2^n(x_2) = (F_1 \times F_2)^n\,(x_1,x_2) &\subseteq (F_1 \times F_2)^n(K) \\ &\subseteq (F_1 \times F_2)^n(\pi_1(K) \times \pi_2(K)) \\ &= F_1^n(\pi_1(K)) \times F_2^n(\pi_2(K)). \end{aligned}$$

Since $x_k \in \pi_k(K) \subseteq U_k$, $k = 1, 2$, by applying to the above inclusion the standard properties of Vietoris limits (cf. [13, 14]) and the squeezing argument, we can finish the proof. ∎

The above considerations allow us to formulate

THEOREM 3.3. *The split square* $\mathbb{Q}$ *is a strict attractor of the IFS* $(\mathbb{Q}; f_{ij} : i, j \in \{0,1\})$,

$$f_{ij}(x,y,t) := \left(\frac{x+i}{2}, \frac{y+j}{2}, t\right), \quad (x,y,t) \in \mathbb{Q}.$$

Proof. Clearly $f_{ij} = g_i \times g_j \times \mathrm{id}$, $i, j \in \{0,1\}$, where the maps g_i are defined in (3.2). According to Remark 2.4, our IFS can be identified with the product of two copies of the IFS from Example 3.1. The application of Lemma 3.2 completes the proof. ∎

4. Split carpet. In this section we construct the split carpet $\mathbb{SC}$, an analogue of the Sierpiński carpet $\mathbb{C}$ in the split square $\mathbb{Q}$, and show that $\mathbb{SC}$ is an attractor of an IFS on $\mathbb{Q}$.

Let us recall that the *Sierpiński carpet* $\mathbb{C}$ is the intersection $\mathbb{C} = \bigcap_{k\in\mathbb{N}} C_k$ of the following sequence of nonempty compact sets

$$\begin{aligned} C_0 &:= [0,1]^2, \\ C_k &:= C_{k-1} \setminus \bigcup_{0\le l,m<3^{k-1}} \left(\frac{3l+1}{3^k}, \frac{3l+2}{3^k}\right) \times \left(\frac{3m+1}{3^k}, \frac{3m+2}{3^k}\right), \ k \ge 1. \end{aligned}$$

Moreover, as is well known, $\mathbb{C}$ is the attractor with full basin of the following IFS:

$$\mathcal{G} = ([0,1]^2; g_{ij} : (i,j) \in \{0,1,2\}^2 \setminus \{(1,1)\}), \tag{4.1}$$

where $g_{ij} : [0,1]^2 \to [0,1]^2$ and

$$g_{ij}(x,y) := \left(\frac{x+i}{3}, \frac{y+j}{3}\right) \quad \text{for } (x,y) \in [0,1]^2. \tag{4.2}$$

Define inductively a nested sequence of nonempty compact sets

$$\begin{aligned} SC_0 &:= Q(0,1;0,1), \\ SC_k &:= SC_{k-1} \setminus \bigcup_{0\le l,m<3^{k-1}} Q\left(\frac{3l+1}{3^k}, \frac{3l+2}{3^k}; \frac{3m+1}{3^k}, \frac{3m+2}{3^k}\right), \ k \ge 1. \end{aligned}$$

The sets SC_k are formed by cutting off from $\mathbb{Q}$ split squares Q similarly as it is done in the construction of the Sierpiński carpet, see Fig. 2. The *split carpet* $\mathbb{SC}$ is the intersection $\mathbb{SC} := \bigcap_{k\in\mathbb{N}} SC_k$.

It should be remarked that $\mathbb{SC}$ is non-metrizable because it contains a copy of the split interval: $\mathbb{SC} \cap \big([0,1] \times \{0\} \times \{3,4\}\big)$; see Fig. 2.

The following simple lemma expresses a strong affinity between $\mathbb{SC}$ and $\mathbb{C}$.

LEMMA 4.1. *Let* $0 \le a < b \le 1$, $0 \le c < d \le 1$. *If*

$$Q(a,b;c,d) \cap \mathbb{SC} \ne \emptyset, \tag{4.3}$$

then $\big((a,b) \times (c,d)\big) \cap \mathbb{C} \ne \emptyset$.

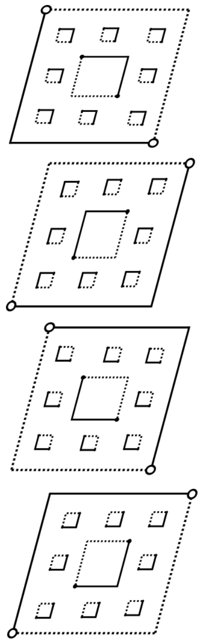

Figure 2. The second generation SC_2 of the sequence SC_k approximating the split carpet $\mathbb{SC}$.

Proof. Suppose that $U \cap \mathbb{C} = \emptyset$, where $U := (a,b) \times (c,d)$. Since U is a connected set, there must be a hole in $\mathbb{C}$ which contains U. That is $U \subseteq (a',b') \times (c',d')$ for some $a' = (3l+1) \cdot 3^{-k}$, $b' = (3l+2) \cdot 3^{-k}$, $c' = (3m+1) \cdot 3^{-k}$, $d' = (3m+2) \cdot 3^{-k}$, where $0 \le l, m < 3^{k-1}$ and $k \ge 1$. Hence $Q(a,b;c,d)$ is contained in one of the holes cut off from $\mathbb{Q}$ in the construction of $\mathbb{SC}$. Namely, $Q(a,b;c,d) \subseteq Q(a',b';c',d')$ as $a' \le a < b \le b'$, $c' \le c < d \le d'$. Since $Q(a',b';c',d') \cap \mathbb{SC} = \emptyset$, this contradicts (4.3). ∎

Now we define an IFS on $\mathbb{Q}$ whose attractor is $\mathbb{SC}$. Let $f_{ij} : \mathbb{Q} \to \mathbb{Q}$, $i,j \in \{0,1,2\}$,

$$f_{ij}(x,y,t) := \left(\frac{x+i}{3}, \frac{y+j}{3}, t\right) \quad \text{for } (x,y,t) \in \mathbb{Q}.$$

Theorem 4.2. *The split carpet $\mathbb{SC}$ is a strict attractor with full basin of the IFS*

$$\mathcal{F} = (\mathbb{Q}; f_{ij} : (i,j) \in \{0,1,2\}^2 \setminus \{(1,1)\}).$$

Proof. Fix $K \in \mathcal{K}(\mathbb{Q})$. We shall show that

$$F^n(K) \to \mathbb{SC} \tag{4.4}$$

in the lower and upper Vietoris topology.

First observe that $F^n(\mathbb{Q}) = SC_n \to \mathbb{SC}$ in the Vietoris sense, because $\mathbb{SC}$ is a descending intersection of nonempty and compact sets SC_n, cf. [14, Sect.I.4, Theorems 4.4 and 4.6, Exercise 4.16]. In particular, we have the upper Vietoris convergence in (4.4).

To verify the lower Vietoris convergence in (4.4), by the squeezing argument, it is enough to pick anyhow $(x,y,t) \in K$ and show the lower Vietoris convergence of $F^n(x,y,t) \to \mathbb{SC}$.

Let $V \subseteq \mathbb{Q}$ be an arbitrary open set with $V \cap \mathbb{SC} \neq \emptyset$. Then $Q(a,b;c,d) \cap \mathbb{SC} \neq \emptyset$ and $Q(a,b;c,d) \subseteq V$ for some $a < b$, $c < d$. Hence, by Lemma 4.1, we have that $U \cap \mathbb{C} \neq \emptyset$ where $U := (a,b) \times (c,d)$. Since $\mathbb{C}$ is the attractor of the IFS $\mathcal{G}$ given by (4.1), (4.2), there exists $n_0 \in \mathbb{N}$ such that $U \cap G^n(x,y) \neq \emptyset$ for $n \geq n_0$. Therefore

$$V \cap F^n(x,y,t) \supseteq U \times \{t\} \cap G^n(x,y) \times \{t\} \neq \emptyset$$

for $n \geq n_0$, due to the relation $F^n(x,y,t) = G^n(x,y) \times \{t\}$. ∎

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